Integrals-tasks (VI part)

Integration of some irrational functions

When we work on rational functions, we have seen that, in principle, we can integrate any rational function.

When we get irrational function, our task is to use substitute or other methods to reduce this integral to the integral of rational functions.

You will learn three methods to solve integrals of irrational functions:

i) **Solving the integral type** $\int R[x,(\frac{ax+b}{cx+d})^{\frac{m}{n}},.......(\frac{ax+b}{cx+d})^{\frac{r}{s}}]dx$

ii) **Integration of differential binomial**

iii) **Substitutes of Euler**

Integral type: $\int R[x,(\frac{ax+b}{cx+d})^{\frac{m}{n}},.......(\frac{ax+b}{cx+d})^{\frac{r}{s}}]dx$

Here we take substitute $\frac{ax+b}{cx+d}=t^k$**, Where** ***k*** **is the smallest common denominator for fractions** $\frac{m}{n},.......,\frac{r}{s}$

$\boxed{\text{Example 1.}}$ $\int \frac{dx}{1+\sqrt{x}}=?$

Here we have only $\sqrt{x}=x^{\frac{1}{2}}$ and substitute will be $x=t^2$.

$$\int \frac{dx}{1+\sqrt{x}}=\left|\begin{matrix} x=t^2 \\ dx=2tdt \end{matrix}\right|=\int \frac{2tdt}{1+t}=2\int \frac{tdt}{1+t}=\text{add and subtract } 1 \text{ in the numerator}=$$

$$2\int \frac{t+1-1}{1+t}dt=2\left(\int \frac{\cancel{t+1}}{\cancel{1+t}}dt-\int \frac{1}{1+t}dt\right)=2\left(\int dt-\int \frac{1}{1+t}dt\right)=2(t-\ln|1+t|)+C\to$$

$$=\boxed{2\sqrt{x}-2\ln\left|1+\sqrt{x}\right|+C}$$

$\boxed{\text{Example 2.}}$ $\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx = ?$

Now we have two different roots $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}} \rightarrow substitute:\ x = t^6,$ because 6 is lowest common denominator of 2 and 3.

$$\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx = \left| \begin{matrix} x = t^6 \\ dx = 6t^5 dt \end{matrix} \right| = \int \frac{\sqrt{t^6}}{1-\sqrt[3]{t^6}} 6t^5 dt = 6\int \frac{t^3}{1-t^2} \cdot t^5 dt = 6\int \frac{t^8}{1-t^2}dt$$

We got a rational function, which was our goal!

$$\frac{t^8}{1-t^2} = -\frac{t^8}{t^2-1}$$

$\cancel{t^8} : (t^2-1) = t^6 + t^4 + t^2 + 1$

$\underline{\pm\cancel{t^8} \mp t^6}$

$\cancel{t^6}$

$\underline{\pm\cancel{t^6} \mp t^4}$

$\cancel{t^4}$

$\underline{\pm\cancel{t^4} \mp t^2}$

$\cancel{t^2}$

$\underline{\pm\cancel{t^2} \mp 1}$

$1 \rightarrow rest$

Then is : $\frac{t^8}{1-t^2} = -\frac{t^8}{t^2-1} = -(t^6 + t^4 + t^2 + 1 + \frac{1}{t^2-1})$

$$6\int \frac{t^8}{1-t^2}dt = -6\int (t^6 + t^4 + t^2 + 1 + \frac{1}{t^2-1})dt = -6(\frac{t^7}{7} + \frac{t^5}{5} + \frac{t^3}{3} + t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|) + C \rightarrow \text{ back in substitute } t = \sqrt[6]{x}$$

$$= \boxed{-6(\frac{\sqrt[6]{x}^7}{7} + \frac{\sqrt[6]{x}^5}{5} + \frac{\sqrt[6]{x}^3}{3} + \sqrt[6]{x} + \frac{1}{2}\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|) + C}$$

Integration of differential binomial

Under this class of integrals we mean forms of integrals $\boxed{\int x^m (a+bx^n)^p \, dx}$.

Expression behind $\int$ is called a *differential binomial.*

We need to "pack" integral in the form in which we can read values for ***m,n,p*** and then for ***a*** and ***b***.

Depending on the values of these numbers, we distinguish **three** situations:

1) If p is an integer, then lifting the binomial $(a+bx^n)$ on p-th degree, this integral is the integral of rational function
2) If $\frac{m+1}{n} \rightarrow$ integer, substitute is $a+bx^n = t^s$, where s is denominator of a fraction p.
3) If $\frac{m+1}{n} + p \rightarrow$ integer, substitute is $ax^{-n} + b = t^s$, where s is denominator of a fraction p.

After substitute, the integral is reduced to integrals of rational functions as we have already mentioned at the beginning of the file.

$\boxed{\text{Example 3.}}$ $\int \sqrt{\frac{x}{1+x^3}} dx = ?$

First, let's "pack" function :

$$\int \sqrt{\frac{x}{1+x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \int \frac{x^{\frac{1}{2}}}{(1+x^3)^{\frac{1}{2}}} dx = \int x^{\frac{1}{2}} \cdot (1+x^3)^{-\frac{1}{2}} dx$$

This compares with integral $\int x^m (a+bx^n)^p \, dx$

$m = \frac{1}{2}; n = 3; p = -\frac{1}{2}$ and $a = 1; b = 1$

$$\frac{m+1}{n} = \frac{\frac{1}{2}+1}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2} \rightarrow \text{ is not integer!}$$

$$\frac{m+1}{n} + p = \frac{1}{2} - \frac{1}{2} = 0 \rightarrow \text{ integer !}$$

So, this is the third situation. We take appropriate substitute:

$$a\cdot x^{-n}+b=t^{s}$$

$m=\dfrac{1}{2};n=3;p=-\dfrac{1}{2}$ and $a=1;b=1$

$a\cdot x^{-n}+b=t^{s}\rightarrow 1\cdot x^{-3}+1=t^{2}\rightarrow \boxed{x^{-3}+1=t^{2}}$ is substitute

Return to the integral. Now we need a good mathematical "technique ".

$$\int\sqrt{\frac{x}{1+x^{3}}}dx=\int x^{\frac{1}{2}}\cdot(1+x^{3})^{-\frac{1}{2}}dx=\left|\begin{array}{l} x^{-3}+1=t^{2}\rightarrow\dfrac{1+x^{3}}{x^{3}}=t^{2}\rightarrow 1+x^{3}=t^{2}x^{3}\rightarrow x^{3}=\dfrac{1}{t^{2}-1}\\ -3x^{-4}dx=2tdt\rightarrow dx=\dfrac{2tdt}{-3x^{-4}}\rightarrow dx=\dfrac{x^{4}2tdt}{-3}\end{array}\right|=$$

$$\int\sqrt{\frac{\cancel{x}}{t^{2}x^{\cancel{3}}}}\frac{x^{4}2tdt}{-3}=\int\frac{1}{\cancel{t}\,\cancel{x}}\frac{x^{\cancel{4}}2\cancel{t}dt}{-3}=-\frac{2}{3}\int x^{3}dt=-\frac{2}{3}\int\frac{1}{t^{2}-1}dt=$$

$$=-\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{t-1}{t+1}\right|+C=-\frac{1}{3}\ln\left|\frac{t-1}{t+1}\right|+C\rightarrow substitute\ \ back$$

$$x^{-3}+1=t^{2}\rightarrow t=\frac{\sqrt{1+x^{3}}}{\sqrt{x^{3}}}$$

$$-\frac{1}{3}\ln\left|\frac{t-1}{t+1}\right|+C=-\frac{1}{3}\ln\left|\frac{\dfrac{\sqrt{1+x^{3}}}{\sqrt{x^{3}}}-1}{\dfrac{\sqrt{1+x^{3}}}{\sqrt{x^{3}}}+1}\right|+C=-\frac{1}{3}\ln\left|\frac{\dfrac{\sqrt{1+x^{3}}-\sqrt{x^{3}}}{\cancel{\sqrt{x^{3}}}}}{\dfrac{\sqrt{1+x^{3}}+\sqrt{x^{3}}}{\cancel{\sqrt{x^{3}}}}}\right|+C=\boxed{-\frac{1}{3}\ln\left|\frac{\sqrt{1+x^{3}}-\sqrt{x^{3}}}{\sqrt{1+x^{3}}+\sqrt{x^{3}}}\right|+C}$$

$\boxed{\text{Example 4.}}$ $\displaystyle\int\frac{dx}{x^{3}\cdot\sqrt[3]{(1+x^{-1})}}=?$

Let's "pack" function and find values for ***m,n,p*** and for ***a*** and ***b***.

$$\int \frac{dx}{x^3 \cdot \sqrt[3]{(1+x^{-1})}} = \int x^{-3} \cdot (1+x^{-1})^{-\frac{1}{3}} dx \rightarrow \int x^m (a+bx^n)^p dx \rightarrow m=-3; n=-1; p=-\frac{1}{3}; \text{ and } a=1; b=1$$

$$\frac{m+1}{n} = \frac{-3+1}{-1} = 2 \rightarrow \boxed{\text{integer}}$$

This is second situacion:

Substitute is : $a+bx^n = t^s \rightarrow \boxed{1+x^{-1}=t^3}$

$$\left| \begin{array}{l} 1+x^{-1}=t^3 \rightarrow x = \dfrac{1}{t^3-1} \\ -x^{-2}dx = 2tdt \rightarrow dx = -x^2 \cdot 2tdt \end{array} \right|$$

$$\int \frac{dx}{x^3 \cdot \sqrt[3]{(1+x^{-1})}} = \int \frac{-\cancel{x^2} \cdot 2tdt}{x^{\cancel{3}} \cdot \sqrt[3]{t^3}} = -2\int \frac{\cancel{t}dt}{x \cdot \cancel{t}} = -2\int \frac{dt}{x}$$

$$= -2\int \frac{dt}{\dfrac{1}{t^3-1}} = -2\int (t^3-1)dt = -2\left(\frac{t^4}{4}-t\right)+C$$

From $1+x^{-1}=t^3 \rightarrow t^3 = 1+\dfrac{1}{x} \rightarrow \boxed{t=\sqrt[3]{\dfrac{1+x}{x}}}, so:$

$$-2\left(\frac{t^4}{4}-t\right)+C = -2\left(\frac{\sqrt[3]{\left(\dfrac{1+x}{x}\right)^4}}{4} - \sqrt[3]{\frac{1+x}{x}}\right)+C = -2\left(\frac{\dfrac{1+x}{x} \cdot \sqrt[3]{\dfrac{1+x}{x}}}{4} - \sqrt[3]{\frac{1+x}{x}}\right)+C$$

$$= -2\sqrt[3]{\frac{1+x}{x}}\left(\frac{\dfrac{1+x}{x}}{4}-1\right)+C = \boxed{-2\sqrt[3]{\frac{1+x}{x}}\left(\frac{1+x}{4x}-1\right)+C}$$

<u>Substitutes of Euler</u>

Euler substitutes we use for solving the integral form $\int R(x, \sqrt{ax^2+bx+c})dx$

This means that the denominator of the integral is $\sqrt{ax^2+bx+c}$ then + or - a linear polynomial in x.

Look out, type of integrals $\int \dfrac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$ we are solving with substitute $mx+n=\dfrac{1}{t}$ … Do not confuse the material ...

First subtitute

In integral $\int R(x, \sqrt{ax^2+bx+c})dx$ look ax^2+bx+c.

If $\boxed{a>0}$ we take substitute $\boxed{\sqrt{ax^2+bx+c} = \pm\sqrt{a}x+t}$. Will we choose the plus or minus in front $\sqrt{a}$ *depends* on specific task. The procedure is still the same for both characters (for example, that we took plus):

$$\sqrt{ax^2+bx+c} = +\sqrt{a}x+t.......................()^2$$
$$ax^2+bx+c = (\sqrt{a}x+t)^2$$
$$\cancel{ax^2}+bx+c = \cancel{ax^2}+2\sqrt{a}\cdot x\cdot t+t^2 \qquad \text{here express x}$$
$$bx-2\sqrt{a}\cdot x\cdot t = t^2-c$$
$$x(b-2\sqrt{a}\cdot t) = t^2-c$$
$$\boxed{x=\frac{t^2-c}{b-2\sqrt{a}\cdot t}}$$

Now differentiating…

Integral is reduced to the integration of rational functions which is "*by t*"

Second subtitute

If $\boxed{c>0}$, we take substitute $\boxed{\sqrt{ax^2+bx+c} = x\cdot t\pm\sqrt{c}}$. As in the previous case, depending on the task, choose plus or minus in front of $\sqrt{c}$

If you choose plus, for example, then will be:

$$\sqrt{ax^2+bx+c} = x\cdot t+\sqrt{c}........................()^2$$
$$ax^2+bx+\cancel{c} = x^2\cdot t^2+2x\cdot t\cdot\sqrt{c}+\cancel{c}$$
$$ax^2+bx-x^2\cdot t^2-2x\cdot t\cdot\sqrt{c}=0$$
$$x^2(a-t^2)+x(b-2\cdot t\cdot\sqrt{c})=0$$
$$x\cdot[x(a-t^2)+(b-2\cdot t\cdot\sqrt{c})]=0.........................A\cdot B=0 \Leftrightarrow A=0 \vee B=0$$
$$x=0 \vee x(a-t^2)+(b-2\cdot t\cdot\sqrt{c})=0$$
$$x(a-t^2)+b-2\cdot t\cdot\sqrt{c}=0$$
$$x(a-t^2)=2\cdot t\cdot\sqrt{c}-b$$
$$\boxed{x=\frac{2\cdot t\cdot\sqrt{c}-b}{a-t^2}}$$

Now differentiating…

Integral is reduced to the integration of rational functions which is "*by t*"

Third subtitute

This substitute is used as the discriminant for ax^2+bx+c is positive .

Then is $ax^2+bx+c=a(x-x_1)(x-x_2)$.

Substitute is $\boxed{\sqrt{a(x-x_1)(x-x_2)}=(x-x_1)\cdot t}$ or $\boxed{\sqrt{a(x-x_1)(x-x_2)}=(x-x_2)\cdot t}$.

Again, depend on the particular task we use $\boxed{\sqrt{a(x-x_1)(x-x_2)}=(x-x_1)\cdot t}$ or $\boxed{\sqrt{a(x-x_1)(x-x_2)}=(x-x_2)\cdot t}$.

If we take for example:

$$\sqrt{a(x-x_1)(x-x_2)}=(x-x_1)\cdot t.............()^2$$
$$a(x-x_1)(x-x_2)=(x-x_1)^2\cdot t^2$$
$$a(x-x_1)(x-x_2)-(x-x_1)^2\cdot t^2=0$$
$$(x-x_1)[a(x-x_2)-(x-x_1)\cdot t^2]=0\to a(x-x_2)-(x-x_1)\cdot t^2=0$$
$$ax-ax_2-x\cdot t^2+x_1\cdot t^2=0$$
$$ax-x\cdot t^2=ax_2-x_1\cdot t^2$$
$$x(a-t^2)=ax_2-x_1\cdot t^2$$
$$\boxed{x=\frac{ax_2-x_1\cdot t^2}{a-t^2}}$$

So, we get a rational function…

$\boxed{\text{Example 5.}}$ $\int\frac{dx}{x+\sqrt{x^2+x+1}}=?$

First, check whether the square function has the solution:

$$x^2+x+1=0\to x_{1,2}=\frac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2\cdot1}=\frac{-1\pm\sqrt{-3}}{2}\to \text{no real solutions!}$$

Here are *a=1, c=1.* We can take the first or second Euler's substitute.

Suppose we take the first.

$\sqrt{ax^2+bx+c} = \pm\sqrt{a}x+t$ When you replace $a = 1$ we get:

$$\sqrt{x^2+x+1} = \pm x+t$$

To choose plus or minus?

We are looking at the function $x+\sqrt{x^2+x+1}$, It is better to choose - , because:

$$x+\sqrt{x^2+x+1} = x+(-x+t) = \cancel{x} \cancel{-x} + t = t$$

Would not be a mistake to take plus , but it complicates the situation and make our own work more difficult.

$$\sqrt{x^2+x+1} = -x+t \qquad ....()^2$$
$$x^2+x+1 = (-x+t)^2$$
$$\cancel{x^2}+x+1 = \cancel{x^2}-2tx+t^2$$
$$x+2tx = t^2-1$$
$$x(1+2t) = t^2-1$$
$$\boxed{x = \frac{t^2-1}{2t+1}} \qquad \text{differentiating...}$$

$$dx = \left(\frac{t^2-1}{2t+1}\right)^{`} dt$$
$$dx = \frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}\,dt$$
$$dx = \frac{4t^2+2t-2t^2+2}{(2t+1)^2}\,dt$$
$$dx = \frac{2t^2+2t+2}{(2t+1)^2}\,dt$$
$$\boxed{dx = \frac{2(t^2+t+1)}{(2t+1)^2}\,dt}$$

We return to the integral:

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \int \frac{\frac{2(t^2+t+1)}{(2t+1)^2}\,dt}{x-x+t} = \int \frac{\frac{2(t^2+t+1)}{(2t+1)^2}\,dt}{t} = \int \frac{2(t^2+t+1)}{t\cdot(2t+1)^2}\,dt = 2\int \frac{t^2+t+1}{t\cdot(2t+1)^2}\,dt$$

We got a rational function. The process of its solution is explained in detail in one of the previous files…

$$\frac{t^2+t+1}{t\cdot(2t+1)^2}=\frac{A}{t}+\frac{B}{2t+1}+\frac{C}{(2t+1)^2}\text{..............................}/\bullet t\cdot(2t+1)^2$$

$$t^2+t+1=A(2t+1)^2+Bt(2t+1)+Ct$$

$$t^2+t+1=A(4t^2+4t+1)+2Bt^2+Bt+Ct$$

$$t^2+t+1=4At^2+4At+A+2Bt^2+Bt+Ct$$

$$t^2+t+1=t^2(4A+2B)+t(4A+B+C)+A$$

$compare:$

$4A+2B=1$

$4A+B+C=1$

$\underline{A=1}$

$$4+2B=1\to 2B=-3\to \boxed{B=-\frac{3}{2}}$$

$$4-\frac{3}{2}+C=1\to \boxed{C=-\frac{3}{2}}$$

$go\ back:$

$$\frac{t^2+t+1}{t\cdot(2t+1)^2}=\frac{1}{t}+\frac{-\frac{3}{2}}{2t+1}+\frac{-\frac{3}{2}}{(2t+1)^2}$$

$$\int\frac{t^2+t+1}{t\cdot(2t+1)^2}dt=\int\frac{1}{t}dt-\frac{3}{2}\int\frac{1}{2t+1}dt-\frac{3}{2}\int\frac{dt}{(2t+1)^2}$$

$$\int\frac{dt}{(2t+1)^2}=\left|\begin{array}{l}2t+1=z\\2dt=dz\\dt=\frac{1}{2}dz\end{array}\right|=\int\frac{\frac{1}{2}dz}{z^2}=\frac{1}{2}\int z^{-2}dz=\frac{1}{2}\frac{z^{-1}}{-1}=-\frac{1}{2z}=\boxed{-\frac{1}{2(2t+1)}}$$

Now is:

$$\int \frac{t^2+t+1}{t\cdot(2t+1)^2}dt = \int \frac{1}{t}dt - \frac{3}{2}\int \frac{1}{2t+1}dt - \frac{3}{2}\int \frac{dt}{(2t+1)^2}$$
$$= \ln|t| - \frac{3}{2}\cdot\frac{1}{2}\ln|2t+1| - \frac{3}{2}\left(-\frac{1}{2(2t+1)}\right) + C$$
$$= \ln|t| - \frac{3}{4}\ln|2t+1| + \frac{3}{4(2t+1)} + C$$

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = 2\int \frac{t^2+t+1}{t\cdot(2t+1)^2}dt = 2\left(\ln|t| - \frac{3}{4}\ln|2t+1| + \frac{3}{4(2t+1)}\right) + C$$

From $\sqrt{x^2+x+1} = -x+t \to \boxed{t = x+\sqrt{x^2+x+1}}$

$$2\left(\ln|t| - \frac{3}{4}\ln|2t+1| + \frac{3}{4(2t+1)} + C\right) =$$
$$\boxed{2\left(\ln\left|x+\sqrt{x^2+x+1}\right| - \frac{3}{4}\ln\left|2(x+\sqrt{x^2+x+1})+1\right| + \frac{3}{4(2(x+\sqrt{x^2+x+1})+1)}\right) + C}$$